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THE EXISTENCE OF INVARIANT PROBABILITY MEASURES FOR A GROUP OF TRANSFORMATIONS

BY

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ABSTRACT

Suppose G is a group of measurable transformations of a σ-finite measure space $(X, \mathcal{A}, \mathrm{m})$. A set $A \in \mathcal{A}$ is weakly wandering under G if there are elements $g_n \in G$ such that the sets $g_n A$, $n = 0, 1, \ldots$, are pairwise disjoint. We prove that the non-existence of any set of positive measure which is weakly wandering under G is a necessary and sufficient condition for the existence of a G-invariant, probability measure defined on $\mathcal{A}$ and dominating the measure m in the sense of absolute continuity.

1. Introduction

Two lines of work concerning the existence of invariant measures lead to the results of this paper.

A) In ergodic theory the problem is formulated as follows.

Given a group G of measurable transformations of a σ-finite measure space $(X, \mathcal{A}, m)$, find necessary and sufficient conditions for the existence of a G-invariant, probability measure μ defined on $\mathcal{A}$ and dominating m in the sense of absolute continuity (see [4, p. 81]). It is usually assumed that the measure m

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is G-quasi-invariant or, equivalently, that every transformation $g \in G$ is nonsingular. The measure μ is then required to be equivalent with m.

In this setup the problem was solved by Hopf [5] for the case when G is a cyclic group and then generalized by Hajian and Ito [3] to arbitrary groups of nonsingular transformations. Hopf's condition states that X is not equivalent by countable decomposition into measurable pieces with any measure theoretically proper subset of itself. The condition of Hajian and Ito excludes the existence of weakly wandering sets of positive measure.

B) A more general approach to the problem is to assume that only a group G of bijections of a set X and a G-invariant σ-algebra $\mathcal{A}$ of subsets of X are given, and then search for "purely combinatorial" conditions for the existence of an *arbitrary* G-invariant probability measure μ defined on $\mathcal{A}$ (see [10, p. 136]).

Tarski found such conditions in the case when μ is only required to be *finitely* instead of countably additive (see [9, §16]). Chuaqui [2], who was apparently unaware of Hajian–Ito's paper, tried to generalize Tarski's results to the countably additive context. In particular, using completely different ideas, he rediscovered a version of the Hajian–Ito theorem.

It is worth noting that even in relatively simple cases when $\mathcal{A}$ is, say, countably generated and contains all singletons, it may still happen that it carries no countably additive, probability, nonatomic measure at all, even without the additional requirement of G-invariance. So, natural necessary conditions for the existence of an invariant measure have the form: "$\mathcal{A}$ carries a measure m satisfying some additional properties" (however, see Section 4). In the case of Chuaqui's theorem these consist of Hopf's condition and the requirement that m is G-quasi-invariant.

Chuaqui (quoted in [10, Question 9.13, p. 137]) conjectured that the latter may be dropped. In [11] it was proved that this can be done in the case when $\mathcal{A}$ is the σ-algebra of all subsets of X.

The aim of the present paper is to prove that Chuaqui's conjecture is true for arbitrary σ-algebras and to generalize at the same time also the Hajian–Ito theorem . It is showed that if a G-invariant σ-algebra $\mathcal{A}$ carries a σ-finite measure m satisfying Hopf's condition (in fact, Hajian–Ito's condition suffices), then there exists a G-invariant, probability measure defined on $\mathcal{A}$ and dominating m (Theorem 3.1).

2. Definitions and preliminaries

Suppose that G is a group of bijections of a set X.

A σ-algebra $\mathcal{A}$ of subsets of X is called G-invariant if $gA \in \mathcal{A}$ whenever $g \in G$ and $A \in \mathcal{A}$, where gA denotes the image of A under g. If m is a (countably additive, non-negative, non-zero) σ-finite measure defined on $\mathcal{A}$ (i.e. $(X, \mathcal{A}, m)$ is a σ-finite measure space) and $\mathcal{A}$ is G-invariant, then we say that G is a group of measurable transformations of the space $(X, \mathcal{A}, m)$.

Suppose that $\mathcal{A}$ is a G-invariant σ-algebra of subsets of X.

Two sets $A, B \in \mathcal{A}$ are called countably G-equidecomposable in $\mathcal{A}$, $A \sim_\infty B$, if there is a partition of A into countably many sets $A_n \in \mathcal{A}$, $n \in \mathbb{N} = \{0, 1, \ldots\}$, and elements $g_n \in G$ such that the sets $g_n A_n$ form a partition of B.

Following [2] we call a set $A \in \mathcal{A}$ G-negligible if X contains pairwise disjoint subsets $A_n \in \mathcal{A}$, $n \in \mathbb{N}$, with each $A_n \sim_\infty A$.

Let N_G consist of all G-negligible elements of $\mathcal{A}$.

We call a set $A \in \mathcal{A}$ weakly wandering under G if there exist elements $g_n \in G$, $n \in \mathbb{N}$, such that $g_n A \cap g_m A = \emptyset$ for $n \neq m$.

Let W_G consist of all elements of $\mathcal{A}$ weakly wandering under G.

Clearly, $W_G \subseteq N_G$.

Suppose that m is a measure on $\mathcal{A}$.

Sets in $\mathcal{A}$ will be called measurable. If μ is another measure on $\mathcal{A}$, then we say that m is absolutely continuous with respect to μ, $m \ll \mu$, if $m(A) = 0$ whenever $\mu(A) = 0$ for all $A \in \mathcal{A}$. If $m \ll \mu$ and $\mu \ll m$, we say that m is equivalent with μ, $m \equiv \mu$.

We say that the measure m is G-quasi-invariant on a set $Z \in \mathcal{A}$, if $m(A) = 0$ implies $m(gA) = 0$ for every $A \in \mathcal{A}$, $g \in G$ such that $A \subseteq Z$ and $gA \subseteq Z$. We say that m is G-quasi-invariant if it is G-quasi-invariant on X. We say that m is G-invariant if $m(gA) = m(A)$ for every $g \in G$ and $A \in \mathcal{A}$.

A set $I \subseteq \mathcal{A}$ is called a σ-ideal in $\mathcal{A}$ if it is closed under countable unions and taking subsets in $\mathcal{A}$. In particular, if ν is a measure on $\mathcal{A}$, then the collection $I_\nu = \{A \in \mathcal{A} : \nu(A) = 0\}$ of all ν -null sets is a σ-ideal in $\mathcal{A}$, called the σ-ideal of the measure ν.

A σ-ideal I in $\mathcal{A}$ is called σ-saturated in $\mathcal{A}$ if there is no uncountable family of pairwise disjoint sets in $\mathcal{A} \setminus I$.

It is well known that if I is the σ-ideal of a σ-finite measure on $\mathcal{A}$, then it is σ-saturated in $\mathcal{A}$.

We shall need the following folklore-like result:

PROPOSITION 2.1: *Suppose that I and J are σ-ideals in $\mathcal{A}$, $I \subseteq J$ and I is σ-saturated in $\mathcal{A}$.*

Then there exists a set $Y \in \mathcal{A}$ such that $X \setminus Y \in J$ and for every $E \in \mathcal{A}, E \in J$ iff $E \cap Y \in I$.

Proof: Let $\mathcal{K}$ be a maximal pairwise disjoint family of sets in $J \setminus I$. Since I is σ-saturated, $\mathcal{K}$ is countable. Hence $\bigcup \mathcal{K} \in J$ and it suffices to let $Y = X \setminus \bigcup \mathcal{K}$. ∎

For $Z \in \mathcal{A}$ let $I_G(Z)$ be the σ-ideal in $\mathcal{A}$ defined by:

$$I_G(Z) = \{A \in \mathcal{A} : m(gA \cap Z) = 0 \text{ for every } g \in G\}.$$

It will later be useful to refer to the following easy observation.

PROPOSITION 2.2: *If G is a countable group of measurable transformations of a σ-finite measure space $(X, \mathcal{A}, m)$ and $m(Z) > 0$, then $I_G(Z)$ is the σ-ideal of a G-quasi-invariant probability measure m' on $\mathcal{A}$. In particular, $I_G(Z)$ is σ-saturated in $\mathcal{A}$.*

Proof: Without loss of generality assume that $m(Z) = 1$. Let $G = \{g_n : n \in \mathbb{N}\}$ and define

$$m'(A) = \sum_{n \in \mathbb{N}} \frac{1}{2^{n+1}} m(g_n A \cap Z). \quad ∎$$

The existence of a G-invariant probability measure μ on $\mathcal{A}$ with $m \ll \mu$ clearly implies that m vanishes on all G-negligible sets in $\mathcal{A}$. The latter property of m is a possible formulation of Hopf's condition mentioned in Section 1. Hajian and Ito [3] proved that the seemingly weaker condition of the non-existence of sets of positive m-measure weakly wandering under G already suffices for the existence of a G-invariant, probability measure $\mu \equiv m$, *provided* that m is G-quasi-invariant.

We shall later use the following consequence of Hajian-Ito's result.

PROPOSITION 2.3: *Let G be a group of measurable transformations of a σ-finite measure space $(X, \mathcal{A}, m)$ and suppose that m vanishes on all measurable sets weakly wandering under G. Then it vanishes also on all measurable G-negligible sets.*

Proof: Take an arbitrary $A \in N_G$. By the definition of $\sim_\infty$, there is a countable subgroup H of G such that $A \in N_H$. Let m' be a H-quasi-invariant, probability measure on $\mathcal{A}$ with $I_{m'} = I_H(X)$, whose existence is guaranteed by Proposition 2.2.

Note that $W_G \subseteq I_m$ implies $W_H \subseteq I_{m'}$. So by Hajian–Ito's theorem there exists a H-invariant probability measure $\mu \equiv m'$. Then since $A \in N_H$, $\mu(A) = 0 = m'(A)$, which in turn implies that $m(A) = 0$. ∎

Hajian–Ito's paper [3] contains an extensive bibliography of the subject. Two more, particularly elegant proofs of the implication "m is G-quasi-invariant and $N_G \subseteq I_m \rightarrow$ there exists a G-invariant probability measure $\mu \equiv m$" can be found in [8] and [7].

3. The main results

Our main result frees Hajian–Ito's theorem from its unnecessary assumption that all transformations are non-singular.

THEOREM 3.1: *Let G be a group of measurable transformations of a σ-finite measure space $(X, \mathcal{A}, m)$.*

Then the following conditions are equivalent:

(i) *There exists a G-invariant, probability measure μ defined on $\mathcal{A}$ such that $m \ll \mu$.*

(ii) *There does not exist any set of positive measure which is weakly wandering under G.*

Proof: We will concentrate on the proof that (ii) $\rightarrow$ (i).

So assume that $W_G \subseteq I_m$.

This clearly implies that $W_G \subseteq I_G(X)$. Note also that if μ is a measure on $\mathcal{A}$ with $I_\mu = I_G(X)$, then $m \ll \mu$. Hence by Hajian–Ito's theorem, the proof will be completed as soon as we establish the following

CLAIM: *$I_G(X)$ is the σ-ideal of a G-quasi-invariant, σ-finite measure ν on $\mathcal{A}$.*

We split the proof of the claim into a series of lemmas. The first one shows how to construct G-quasi-invariant measures from measures which are G-quasi-invariant on certain specific subsets of X.

LEMMA 3.2: *If m is G-quasi-invariant on a set Z of positive measure and the σ-ideal $I_G(Z)$ is σ-saturated in $\mathcal{A}$, then there exists a G-quasi-invariant, probability measure ν on $\mathcal{A}$ with $I_\nu = I_G(Z)$.*

Proof of Lemma 3.2: Let $\mathcal{K}$ be a maximal collection of measurable, pairwise disjoint sets with the property that for each $D \in \mathcal{K}$ there exists $g \in G$ such that $gD \subseteq Z$ and $0 < m(gD) < \infty$.

Since the σ-ideal $I_G(Z)$ is σ-saturated, $\mathcal{K}$ is countable. So let $\mathcal{K} = \{D_k\}$ and for each k fix $h_k \in G$ with $h_k D_k \subseteq Z$ and $0 < m(h_k D_k) < \infty$.

Define

$$\nu(B) = \sum_k m(h_k[B \cap D_k]) \quad \text{for } B \in \mathcal{A}.$$

Clearly, ν is a measure on $\mathcal{A}$.

To prove the G-quasi-invariance of ν it is enough to show that $I_\nu = I_G(Z)$.

The inclusion "$\supseteq$" is obvious. To see the converse, take an arbitrary $B \in I_\nu$. Note that the maximality of the collection $\mathcal{K}$ implies that $B \setminus \bigcup_k D_k \in I_G(Z)$. Hence it suffices to prove that $B \cap D_k \in I_G(Z)$ for every k.

So fix arbitrary k and $g \in G$.

Note that:

$$g[B \cap D_k] \cap Z \subseteq g h_k^{-1}[h_k[B \cap D_k] \cap h_k g^{-1} Z] \subseteq Z.$$

But $m(h_k[B \cap D_k]) = 0$ and $h_k[B \cap D_k] \subseteq Z$, so by the G-quasi-invariance of m on Z, $m(g[B \cap D_k] \cap Z) = 0$.

Finally, the measure ν is σ-finite since $\nu(X \setminus \bigcup_k D_k) = 0$ and $\nu(D_k) = m(h_k D_k) < \infty$ for every k. To complete the proof replace ν by an equivalent probability measure. ∎

The next lemma gives a useful reformulation of the claim.

LEMMA 3.3: *The following conditions are equivalent:*

(i) *There exists a G-quasi-invariant, σ-finite measure ν on $\mathcal{A}$ such that $I_\nu = I_G(X)$.*

(ii) *The σ-ideal $I_G(X)$ is σ-saturated in $\mathcal{A}$.*

Proof: Obviously, (i) $\to$ (ii). To prove that (ii) $\to$ (i), use Proposition 2.1 with $I = I_G(X)$ and $J = I_m$ to find a set $Y \in \mathcal{A}$ such that $m(X \setminus Y) = 0$ and m is G-quasi-invariant on Y. Notice that $I_G(X) = I_G(Y)$, so the existence of ν immediately follows from Lemma 3.2. ∎

Note that by Proposition 2.3, $N_G \subseteq I_m$. Hence, in view of the preceding lemma, to complete the proof of the claim it suffices to establish the following.

LEMMA 3.4: *If $N_G \subseteq I_m$, then the σ-ideal $I_G(X)$ is σ-saturated in $\mathcal{A}$.*

Proof:* Suppose, towards a contradiction, that there exist uncountably many pairwise disjoint sets $A_t \in \mathcal{A}$ with associated functions $g_t \in G$ such that $m(g_t A_t) > 0$, for each $t \in T$.

Since the measure m is σ-finite, there exists a countable set $T_0 \subseteq T$ such that

$$m(g_t A_t \setminus \bigcup_{s \in T_0} g_s A_s) = 0 \quad \text{for every } t \in T.$$

Let

$$C_0 = \bigcup_{s \in T_0} g_s A_s.$$

Proceed by induction to define measurable sets C_n and countable, pairwise disjoint subsets T_n of T keeping the following conditions satisfied for every $n \in \mathbb{N}$:

(1) $$C_n = \bigcup_{s \in T_n} g_s A_s.$$

(2) $$m(g_t A_t \setminus C_n) = 0 \quad \text{for every } t \in T \setminus \bigcup_{k<n} T_k.$$

Finally pick up an arbitrary $t_0 \in T \setminus \bigcup_{n \in \mathbb{N}} T_n$ and let

$$C_\infty = g_{t_0} A_{t_0} \cap \bigcap_{n \in \mathbb{N}} C_n.$$

By (1), C_∞ is for every $n \in \mathbb{N}$ countably G-equidecomposable in $\mathcal{A}$ with a subset of $\bigcup_{s \in T_n} A_s$.

Hence $C_\infty \in N_G$.

But it easily follows from (2) that $m(C_\infty) > 0$ contradicting the assumption that $N_G \subseteq I_m$. ∎

This completes the proof of the claim.

By the remarks preceding the formulation of the claim, this completes also the proof of Theorem 3.1. ∎

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* I owe this proof to Marcin Penconek who simplified the much more complicated original argument.

Let us note that the above proof gives a G-invariant measure μ which is the least, in the sense of $\ll$, G-invariant, σ-finite measure on $\mathcal{A}$ that dominates the measure m. This is due to the fact that $I_\nu \subseteq I_G(X) = I_\mu$, whenever ν is a G-invariant, σ-finite measure on $\mathcal{A}$ such that $m \ll \nu$.

We close this section with an explicit formulation of the positive answer to Chuaqui's question (see [10, Question 9.13, p. 137]), which follows immediately from Theorem 3.1.

THEOREM 3.5: *Let G be a group of bijections of a set X and $\mathcal{A}$ a G-invariant σ-algebra of subsets of X.*

Then the following conditions are equivalent:

(i) There exists a G-invariant probability measure on $\mathcal{A}$.

(ii) There exists a probability measure on $\mathcal{A}$ that vanishes on N_G. ∎

4. Remarks

Tarski's work on finitely additive invariant measures motivates the question, whether condition (ii) in Theorem 3.5 can be replaced by the seemingly weaker requirement that X is not G-negligible.

Chuaqui conjectured that this is the case but he later found a counterexample (see [10, Theorem 9.12]).

However, the situation is different when only the groups of Borel automorphisms of standard Borel spaces are considered.

Nadkarni [6, 3.1, p.215] proved that if G is the group generated by a single Borel automorphism of a Polish space X, then the condition $X \notin N_G$ is necessary and sufficient for the existence of a G-invariant, probability measure on the σ-algebra $\mathcal{B}$ of Borel subsets of X.

A. S. Kechris (private communication) has pointed out that the above result, extended in a straightforward manner to an arbitrary countable group G of Borel automorphisms of X and then, by continuity, to an an arbitrary Polish group G acting continuously on X, combined with a recent result of H. Becker [1], which implies that Borel actions of Polish groups on Polish spaces are Borel isomorphic to continuous ones, has the following immediate consequence:

If X is a Polish space, G is a Polish group and the function $(g, x) \to gx$ from $G \times X$ to X is Borel measurable, then the following conditions are equivalent:

(i) There exists a G-invariant probability measure on the σ-algebra $\mathcal{B}$ of Borel subsets of X.
(ii) X is not G-negligible. ∎

There is, however, no hope to generalize this still further to arbitrary group G of Borel automorphisms of X, as the following example shows.

PROPOSITION 4.1: *Let G be the group of all Borel automorphisms g of a Polish space X with the property that the set $\{x \in X : gx \neq x\}$ is meager in X.*

Then X is not G-negligible but there is no G-invariant, probability measure on the σ-algebra $\mathcal{B}$ of Borel subsets of X.

Proof: It is not difficult to prove that N_G coincides with the collection of all meager Borel subsets of X. Hence $X \notin N_G$ and if μ was a G-invariant, probability measure on $\mathcal{B}$, then it would vanish on all meager Borel sets. But this contradicts the well-known fact, that every probability measure on $\mathcal{B}$ is concentrated on a meager set. ∎

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